COMP2111 Week 7
Term 1, 2024
State machines

Summary

- Motivation
- Definitions
- The invariant principle
- Partial correctness and termination
- Input and output
- Finite automata

Motivation

In Assignment 1, we modelled programs as relations between initial and final states of successful executions. So this Tic-Tac-Toe program:

```
void move(int pos, char fill) {
    if(board[pos] = "E" && (fill = "X" || fill = "O")) {
        board[pos] := fill;
    }
    else abort;
}
```

can be modelled with this relation:

$$\{(b, b') \ : \ \exists n.\, \forall i.\ (n = i \ \rightarrow \ b_i = \mathrm{E} \wedge b'_i \neq \mathrm{E}) \wedge (n \neq i \rightarrow b_i = b'_i)\}$$

Motivation

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Possibility of failure is sometimes not captured. This:

void incr() {

$x := x + 1;$

}

can be modelled by this relation over $\mathbb{Z} \times \mathbb{Z}$:

$$\{(x, x') \quad : \quad x + 1 = x'\}$$

Motivation

Such relational modelling is useful (spoiler alert: W8-9), but doesn't always capture everything we care about.

Possibility of failure is sometimes not captured. This:

```
void incr() {
   if(Math.random() < .5) then abort else
      x := x + 1;
   }
}
```

can *also* be modelled by this relation over $\mathbb{Z} \times \mathbb{Z}$:

$$\{(x, x') \quad : \quad x + 1 = x'\}$$

Motivation

Such relational modelling is useful (spoiler alert: W8-9), but doesn't always capture everything we care about.

Sometimes the final state isn't what's interesting.

```
void yes() {
  while(true) print("y\n");
}
```

This program has no final states, so its relational model doesn't say much:

$$\{\}$$

Motivation

State machines model step-by-step processes with:

- A set of *states*, possibly including a designated *start state*.
- A *transition relation*, detailing how to move (transition) from one state to another.

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Example

The semantics of a program:

- States: functions from variable names to values
- Transitions: execute a line of code.

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Example

A game of noughts and crosses

- States: Board positions
- Transitions: Legal moves

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Example

Stateful communication protocols: e.g. SMTP

- States: Stages of communication
- Transitions: Determined by commands given (e.g. HELO, DATA, etc)

Motivation

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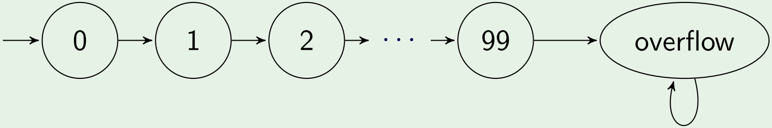
Example

A bounded counter that counts from 0 to 99 and overflows at 100:



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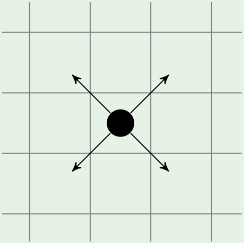
Example

A robot that moves diagonally



States: Locations
Transitions: Moves

Motivation

State machines model step-by-step processes with:

- A set of *states*, possibly including a designated *start state*.
- A *transition relation*, detailing how to move (transition) from one state to another.

Example

Die Hard jug problem: Given jugs of 3L and 5L, measure out exactly 4L.

- States: Defined by amount of water in each jug
- Start state: No water in both jugs
- Transitions: Pouring water (in, out, jug-to-jug)

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Definitions

A **transition system** is a pair $(S, \rightarrow)$ where:

- S is a set (of **states**), and
- $\rightarrow \subseteq S \times S$ is a (**transition**) **relation**.

If $(s, s') \in \rightarrow$ we write $s \rightarrow s'$.

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  - $\rightarrow \subseteq S \times L \times S$
  - $(s, a, s') \in \rightarrow$ is written as $s \xrightarrow{a} s'$
- If $\rightarrow$ is a partial function we say that the system is **deterministic**, otherwise it is **non-deterministic**

Example: Bounded counter

Example

A bounded counter that counts from 0 to 99 and overflows at 100:

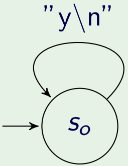
$$\rightarrow (0) \rightarrow (1) \rightarrow (2) \rightarrow \cdots \rightarrow (99) \rightarrow (\text{overflow}) \circlearrowleft$$

- $S = \{0, 1, \ldots, 99, \text{overflow}\}$
- $$\begin{aligned} \rightarrow \;=\; & \{(i, i+1) \;:\; 0 \leq i < 99\} \\ & \cup \{(99, \text{overflow})\} \\ & \cup \{(\text{overflow}, \text{overflow})\} \end{aligned}$$
- $s_0 = 0$
- Deterministic

Example: yes

Example

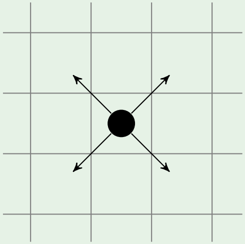
```
void yes() {
    while(true) print("y\n");
}
```



$S = \{s_0\}$
$L =$ the set of strings
$s_0 \xrightarrow{\text{"y\textbackslash n"}} s_0$

Example: Diagonally moving robot

Example



States: Locations

Transitions: Moves

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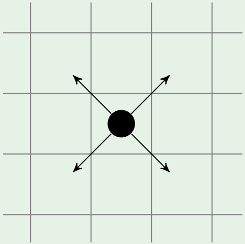


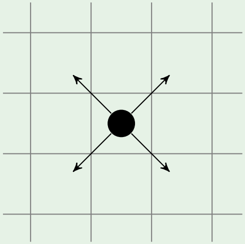
$S = \mathbb{Z} \times \mathbb{Z}$

$(x, y) \to (x \pm 1, y \pm 1)$

Non-deterministic

Example: Diagonally moving robot

Example



$$S = \mathbb{Z} \times \mathbb{Z}$$

$$L = \{NW, NE, SW, SE\}$$

$$(x, y) \xrightarrow{NW} (x-1, y+1)$$

$$(x, y) \xrightarrow{NE} (x+1, y+1)$$

$$(x, y) \xrightarrow{SW} (x-1, y-1)$$

$$(x, y) \xrightarrow{SE} (x+1, y-1)$$

Deterministic

Example: Die Hard jug problem

Example

Given jugs of 3L and 5L, measure out exactly 4L.

- States: Defined by amount of water in each jug
- Start state: No water in both jugs
- Transitions: Pouring water (in, out, jug-to-jug)

Example: Die Hard jug problem

Example

Given jugs of 3L and 5L, measure out exactly 4L.

- $S = \{(i,j) \in \mathbb{N} \times \mathbb{N} \ : \ 0 \leq i \leq 5 \text{ and } 0 \leq j \leq 3\}$
- $s_0 = (0,0)$
- $\rightarrow$ given by

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  - $(i,j) \rightarrow (0,j)$ [empty 5L jug]
  - $(i,j) \rightarrow (i,0)$ [empty 3L jug]

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Given jugs of 3L and 5L, measure out exactly 4L.

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- $s_0 = (0, 0)$
- $\rightarrow$ given by
  - $(i, j) \rightarrow (i + j, 0)$ if $i + j \leq 5$ [empty 3L jug into 5L jug]
  - $(i, j) \rightarrow (0, i + j)$ if $i + j \leq 3$ [empty 5L jug into 3L jug]

Example: Die Hard jug problem

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  - $(i,j) \rightarrow (5, j-5+i))$ if $i+j \geq 5$ [fill 5L jug from 3L jug]
  - $(i,j) \rightarrow (i-3+j, 3)$ if $i+j \geq 3$ [fill 3L jug from 5L jug]

Example: Die Hard jug problem

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Given jugs of 3L and 5L, measure out exactly 4L.

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Runs and reachability

Given a transition system $(S, \rightarrow)$ and states $s, s' \in S$,

- a **run** (or **trace**) from s is a (possibly infinite) sequence $s_1, s_2, \ldots$ such that $s = s_1$ and $s_i \rightarrow s_{i+1}$ for all $i \geq 1$.

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- A run is **maximal** if it cannot be extended; i.e., it is either infinite, or ends in a state from which there are no transitions.
- we say s' is **reachable** from s, written $s \to^* s'$, if (s, s') is in the reflexive and transitive closure of $\to$.

NB

s' is reachable from s if there is a run from s which contains s'.

Reachability example: Die Hard jug problem

Example

Given jugs of 3L and 5L, measure out exactly 4L.

- States: $S = \{(i, j) \in \mathbb{N} \times \mathbb{N} \;:\; 0 \leq i \leq 5 \text{ and } 0 \leq j \leq 3\}$
- Transition relation: $(i, j) \to (0, j)$ etc.

Is $(4, 0)$ reachable from $(0, 0)$?

Reachability example: Die Hard jug problem

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Given jugs of 3L and 5L, measure out exactly 4L.

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Is $(4,0)$ reachable from $(0,0)$?

Yes:

$$\begin{array}{ccccc} (0,0) & \rightarrow & (0,3) & \rightarrow & (3,0) \\ & & & & \downarrow \\ (0,1) & \leftarrow & (5,1) & \leftarrow & (3,3) \\ \downarrow & & & & \\ (1,0) & \rightarrow & (1,3) & \rightarrow & (4,0) \end{array}$$

Safety and Liveness

Transition systems can be used to study whether systems satisfy **safety** and **liveness** properties.

Safety: something bad will never happen.

Liveness: something good will happen.

Contrast this with **reachability**:

Reachability: something good can happen.

Safety and Liveness: Examples

Example

Suppose our transition system models a nuclear power plant.

Safety: the reactor never reaches the `meltdown` state.

Liveness: the power plant will keep supplying power.

Safety and Liveness: Examples

Example

```
void yes() {
    while(true){
        print("y\n");
    }
}
```

Safety: *yes*() never prints anything but "y\n".

Liveness: *yes*() will always print another "y\n".

Safety and Liveness: Examples

Example

$$
\begin{array}{l}
y := 1; \\
z := x; \\
\text{while}(z \neq 0)\{ \\
\quad y := y * z; \\
\quad z := z - 1; \\
\}
\end{array}
$$

Safety: If the program ever terminates, then $y = x!$

Liveness: The program will terminate

(How is that a safety property?)

Safety and Liveness

A *property* is a set of infinite runs. (Terminating runs can be made infinite by adding a self-loop to the final state.)

Safety: A *safety property* can be falsified by a finite prefix of a behaviour.

Liveness: A *liveness property* can always be satisfied eventually.

Properties Examples

Are they safety or liveness?

- *When I come home, there must be beer in the fridge*

Properties Examples

Are they safety or liveness?

- *When I come home, there must be beer in the fridge* – **Safety**
- *When I come home, I'll drop on the couch and drink a beer*

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- *When I come home, there must be beer in the fridge* – **Safety**
- *When I come home, I'll drop on the couch and drink a beer* – **Liveness**
- *I'll be home later* – **Liveness**
- *The program never allocates more than 100MB of memory*

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- *If a process wishes to enter its critical section, it will eventually be allowed to do so*

Properties Examples

Are they safety or liveness?

- *When I come home, there must be beer in the fridge* – **Safety**
- *When I come home, I'll drop on the couch and drink a beer* – **Liveness**
- *I'll be home later* – **Liveness**
- *The program never allocates more than 100MB of memory* — **Safety**
- *The program will allocate at least 100MB of memory* – **Liveness**
- *No two processes are simultaneously in their critical section* — **Safety**
- *If a process wishes to enter its critical section, it will eventually be allowed to do so* – **Liveness**

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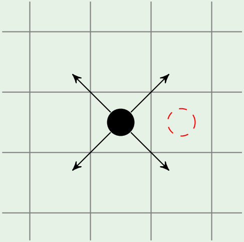
Safety example: Diagonally moving robot

Example



Starting at $(0, 0)$

Can the robot get to $(0, 1)$?

Safety example: Diagonally moving robot

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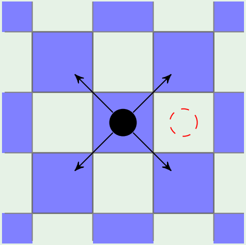

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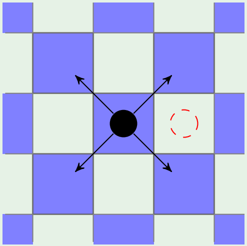

Starting at $(0,0)$

Can the robot get to $(0,1)$? No

Safety example: Diagonally moving robot

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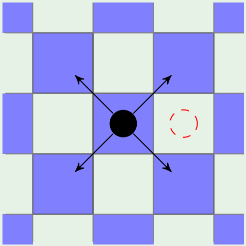

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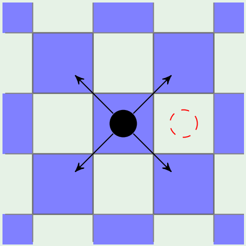
$\text{isBlue}((m, n)) := 2|(m + n)$

Safety example: Diagonally moving robot

Example



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Can the robot get to $(0, 1)$? No

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if $\text{isBlue}(s)$ and $s \to s'$
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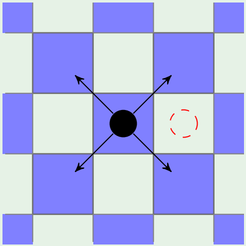

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if $\text{isBlue}(s)$ and $s \to s'$
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$\text{isBlue}((0, 0))$ and $\neg\text{isBlue}((0, 1))$

The invariant principle

A **preserved invariant** of a transition system is a unary predicate φ on states such that if $\varphi(s)$ holds and $s \to s'$ then $\varphi(s')$ holds.

Invariant principle

If a preserved invariant holds at a state s, then it holds for all states reachable from s.

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A **preserved invariant** of a transition system is a unary predicate φ on states such that if $\varphi(s)$ holds and $s \to s'$ then $\varphi(s')$ holds.

Invariant principle

If a preserved invariant holds at a state s, then it holds for all states reachable from s.

Proof sketch: Let s' be a state reachable from s. We can show $\varphi(s')$ by induction on the length of the run from s to s'.

Invariant example: Modified Die Hard problem

Example

Given jugs of 3L and 6L, measure out exactly 4L.

- States: $S = \{(i,j) \in \mathbb{N} \times \mathbb{N} \ : \ 0 \leq i \leq 6 \text{ and } 0 \leq j \leq 3\}$
- Transition relation: $(i,j) \to (0,j)$ etc.

Is $(4,0)$ reachable from $(0,0)$?

Invariant example: Modified Die Hard problem

Example

Given jugs of 3L and 6L, measure out exactly 4L.

- States: $S = \{(i,j) \in \mathbb{N} \times \mathbb{N} \,:\, 0 \le i \le 6 \text{ and } 0 \le j \le 3\}$
- Transition relation: $(i,j) \to (0,j)$ etc.

Is $(4,0)$ reachable from $(0,0)$?
No. Consider $\varphi((i,j)) = (3|i) \wedge (3|j)$.

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Partial correctness

Let $(S, \rightarrow, s_0, F)$ be a transition system with start state s_0 and final states F and a φ be a unary predicate on S. We say the system is **partially correct for** φ if $\varphi(s')$ holds for all states $s' \in F$ that are reachable from s_0.

NB

Partial correctness is a safety property. It doesn't say whether the transition system will ever reach a final state.

Partial correctness example: Fast exponentiation

Example

Consider the following program in $\mathcal{L}$:

$$
\begin{array}{l}
x := m; \\
y := n; \\
r := 1; \\
\text{while } y > 0 \text{ do} \\
\quad \text{if } 2|y \text{ then} \\
\quad\quad y := y/2 \\
\quad \text{else} \\
\quad\quad y := (y-1)/2; \\
\quad\quad r := r * x \\
\quad \text{fi;} \\
\quad x := x * x \\
\text{od}
\end{array}
$$

Partial correctness example: Fast exponentiation

Example

- States: Functions from $\{m, n, x, y, r\}$ to $\mathbb{N}$
- Transitions:

Partial correctness example: Fast exponentiation

Example

- States: $(x, y, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$
- Transitions: Effect of each line of code?

Partial correctness example: Fast exponentiation

Example

- States: $(x, y, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$
- Transitions: Effect of each iteration of while loop
  - $(x, y, r) \to (x^2, y/2, r)$ if y is even
  - $(x, y, r) \to (x^2, (y-1)/2, rx)$ if y is odd

Partial correctness example: Fast exponentiation

Example

- States: $(x, y, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$
- Transitions: Effect of each iteration of while loop
  - $(x, y, r) \rightarrow (x^2, y/2, r)$ if y is even
  - $(x, y, r) \rightarrow (x^2, (y-1)/2, rx)$ if y is odd
- Start state: $(m, n, 1)$

Partial correctness example: Fast exponentiation

Example

- States: $(x, y, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$
- Transitions: Effect of each iteration of while loop
  - $(x, y, r) \to (x^2, y/2, r)$ if y is even
  - $(x, y, r) \to (x^2, (y-1)/2, rx)$ if y is odd
- Start state: $(m, n, 1)$
- Final states: $\{(x, 0, r) \;:\; x, r \in \mathbb{N}\}$

Partial correctness example: Fast exponentiation

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How can we show total correctness?

Total correctness = safety + liveness

A transition system $(S, \rightarrow)$ **terminates** from a state $s \in S$ if all runs from s have finite length.

A transition system is **totally correct for a unary predicate** φ, if it terminates (from s_0) and φ holds in the last state of every run.

Measure

In a transition system $(S, \to)$, a **measure** is a function $f : S \to \mathbb{N}$.

A measure is **strictly decreasing** if $s \to s'$ implies $f(s') < f(s)$.

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A measure is **strictly decreasing** if $s \to s'$ implies $f(s') < f(s)$.

Theorem

If f is a strictly decreasing measure, then the length of any run from s is at most $f(s)$.

Termination example: Fast exponentiation

Example

- States: $(x, y, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$
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Measure:

Termination example: Fast exponentiation

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Measure: $f((x, y, r)) = y$

Summary

- Motivation
- Definitions
- The invariant principle
- Partial correctness and termination
- Input and output
- Finite automata

Interaction with the environment

We can model the system interacting with an external entity via inputs (Σ) and outputs (Γ) by using **labelled transitions**:
$\rightarrow \subseteq S \times L \times S$ where $L = \Sigma \times \Gamma$

We'll look at categories of input/output transition systems:

Acceptors: Accept/reject a sequence of inputs

Transducers: Take a sequence of inputs and produce a sequence of outputs

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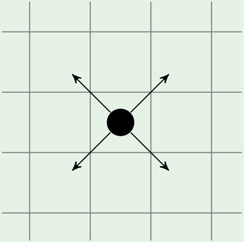
We'll look at categories of input/output transition systems:

Acceptors: Accept/reject a sequence of inputs (Relations)

Transducers: Take a sequence of inputs and produce a sequence of outputs (Functions)

Acceptor example: Diagonally moving robot

Example



$S = \mathbb{Z} \times \mathbb{Z}$

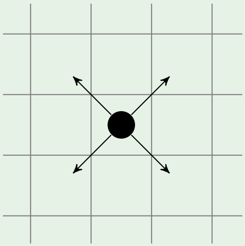
$s_0 = (0, 0)$

$(x, y) \xrightarrow{NW} (x - 1, y + 1)$
$(x, y) \xrightarrow{NE} (x + 1, y + 1)$
$(x, y) \xrightarrow{SW} (x - 1, y - 1)$
$(x, y) \xrightarrow{SE} (x + 1, y - 1)$

Accept if $(2, 2)$ reached

Acceptor example: Diagonally moving robot

Example



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Accepted sequences:
NE, NE

Acceptor example: Diagonally moving robot

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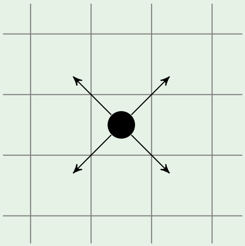

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Accepted sequences:
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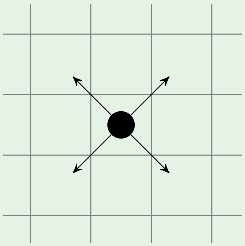
NE, SE, NE, NW

Acceptor example: Diagonally moving robot

Example



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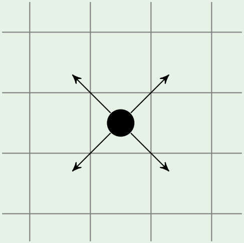
Accept if $(2, 2)$ reached

Accepted sequences:
NE, NE
NE, SE, NE, NW
NE, NE, NE, SW ...

Transducer example: Diagonally moving robot

Example



$S = \mathbb{Z} \times \mathbb{Z}$

$s_0 = (0, 0)$

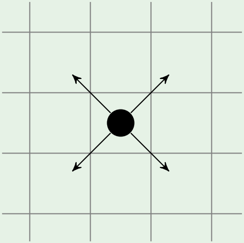
$$(x, y) \xrightarrow{NW/x} (x - 1, y + 1)$$
$$(x, y) \xrightarrow{NE/x} (x + 1, y + 1)$$
$$(x, y) \xrightarrow{SW/x} (x - 1, y - 1)$$
$$(x, y) \xrightarrow{SE/x} (x + 1, y - 1)$$

Input direction

Output x-coordinate

Transducer example: Diagonally moving robot

Example



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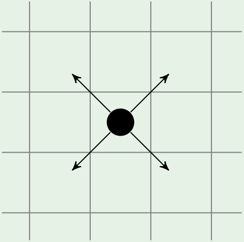
Input direction
Output x-coordinate

Input: NE, SE, NE, NW
Output: $1, 2, 3, 2$

Transducer example: Diagonally moving robot

Example



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$$(x, y) \xrightarrow{SE/y} (x+1, y-1)$$

Input direction

Output y-coordinate

Input: NE, SE, NE, NW

Output: $1, 0, 1, 2$

Acceptor example: Die Hard jug problem

Example

- $S = \{(i,j) \in \mathbb{N} \times \mathbb{N} \;:\; 0 \leq i \leq 5 \text{ and } 0 \leq j \leq 3\}$
- $s_0 = (0,0)$
- $\rightarrow$ given by
  - $(i,j) \xrightarrow{\text{E5}} (0,j)$ [empty 5L jug]
  - $(i,j) \xrightarrow{\text{E3}} (i,0)$ [empty 3L jug]
  - $(i,j) \xrightarrow{\text{F5}} (5,j)$ [fill 5L jug]
  - $(i,j) \xrightarrow{\text{F3}} (i,3)$ [fill 3L jug]
  - $(i,j) \xrightarrow{\text{E35}} (i+j,0)$ if $i+j \leq 5$ [empty 3L jug into 5L jug]
  - $(i,j) \xrightarrow{\text{E53}} (0,i+j)$ if $i+j \leq 3$ [empty 5L jug into 3L jug]
  - $(i,j) \xrightarrow{\text{F53}} (5,j-5+i))$ if $i+j \geq 5$ [fill 5L jug from 3L jug]
  - $(i,j) \xrightarrow{\text{F35}} (i-3+j,3)$ if $i+j \geq 3$ [fill 3L jug from 5L jug]
- Accept if $(4,0)$ is reached:

Acceptor example: Die Hard jug problem

Example

- $S = \{(i,j) \in \mathbb{N} \times \mathbb{N} \;:\; 0 \leq i \leq 5 \text{ and } 0 \leq j \leq 3\}$
- $s_0 = (0,0)$
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  - $(i,j) \xrightarrow{\text{E5}} (0,j)$ [empty 5L jug]
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  - $(i,j) \xrightarrow{\text{F53}} (5,j-5+i))$ if $i+j \geq 5$ [fill 5L jug from 3L jug]
  - $(i,j) \xrightarrow{\text{F35}} (i-3+j,3)$ if $i+j \geq 3$ [fill 3L jug from 5L jug]
- Accept if $(4,0)$ is reached: e.g. F3, E35, F3, F53, E5, E35, F3, E35

ϵ-transitions

It can be useful to allow the system to transition without taking input or producing output. We use the special symbol ϵ to denote such transitions.

Formal definitions

An **acceptor** is a $\Sigma \cup \{\epsilon\}$-labelled transition system $A = (S, \rightarrow, \Sigma, s_0, F)$ with a start state $s_0 \in S$ and a set of final states $F \subseteq S$.

A **transducer** is a $(\Sigma \cup \{\epsilon\}) \times (\Gamma \cup \{\epsilon\})$-labelled transition system $T = (S, \rightarrow, \Sigma, s_0, F)$ with a start state $s_0 \in S$ and a set of final states $F \subseteq S$.

Summary

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Finite state transition systems

State transition systems with a finite set of states are particularly useful in Computer Science.

Acceptors: **Finite state automata**

Transducers: **Mealy machines**